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The problem of the time-optimal control of spacecraft reorientation☆

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ABSTRACT

The use of Pontryagin's maximum principle to solve spacecraft motion control problems is demonstrated. The problem of the optimal control of the spatial reorientation of a spacecraft (as a rigid body) from an arbitrary initial angular position to an assigned final angular position in the minimum rotation time is investigated in detail. The case in which velocity parameters of the motion are constrained is considered. An analytical solution of the problem is obtained in closed form using the method of quaternions, and mathematical expressions for synthesizing the optimal control programme are given. The kinematic problem of spacecraft reorientation is solved completely. A design scheme for solving the maximum principle boundary-value problem for arbitrary turning conditions and inertial characteristics of the spacecraft is given. A solution of the problem of the optimal control of spatial reorientation for a dynamically symmetrical spacecraft is presented in analytical form (to expressions in elementary functions). The results of mathematical modelling of the motion of a spacecraft under optimal control, which confirm the practical feasibility of the control algorithm developed, are given. Estimates have shown that the turn time of modern spacecraft with a constrained magnitude of the angular momentum can be reduced by 15–25% compared with conventional reorientation methods. The greatest effect is achieved for turns through large angles (90° or more) when the final rotation vector is equidistant from the longitudinal axis and the transverse plane of the spacecraft.

The problem of the optimal control of the angular position of a rigid body has been investigated in various formulations in many publications.[1–8] In particular, the kinematic problem of turning has been investigated in detail,[1] and a solution was presented for the version in which the magnitude of the angular velocity vector is constrained. Questions of the optimal turning of a spacecraft for maximum speed and minimum energy consumption have been considered,[2] and an analytical solution has been obtained using Pontryagin's maximum principle for the case in which the region of admissible values of the control moment is confined to a sphere, and the spacecraft itself turns about the final rotation vector. Control by a combined synthesis method based on a generalized work criterion has been devised.[3] Although analytical design based on a generalized work criterion does not require significant simplification of the model of the control object, optimization by this method does not enable the constraints on the control variables to be satisfied. In addition, the method is applicable only when the initial angular deviations are relatively small.[4] The unsuitability of analytical design with respect to a generalized work criterion for the case of arbitrary initial angular deviations (including deviations up to 180°) is a serious deficiency of this method. The use of predicting models to synthesise the controls improves the alignment quality, and they are, therefore, currently widely employed. However, in such algorithms the final result largely depends on the form of the predicting model; the latter completely predetermines the type of turn realized by the controls obtained. Adopting a predicting model that is as close as possible to reality entails unavoidable mathematical complications. For the most part, optimization methods employing predicting models are used to synthesize controls that stabilize the programmed motion of a spacecraft and to design high-precision compensations.[5,6] The possibility of finding similar programmed controls has not been thoroughly investigated. The solution obtained in the latest known publications on this subject[3,5,6] is not fundamentally new. The control created turns the spacecraft about the Euler axis, although the optimization methods and control algorithms are different. At the same time, an Euler turn is not always optimal according to the time criterion (and is optimal only in a few special cases) no matter how precisely it is performed. The optimal turn problem has been solved completely for only two special cases, viz., a flat turn about the principal central axis of inertia of a spacecraft[4,7] and a banked turn of a spherically symmetrical body.[1] The attitude control of large heavy spacecraft has its own special features.[8]

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In all the numerous known publications devoted to this problem, no constraint was imposed on the angular momentum vector for optimizing the control of the angular position of a spacecraft during a banked turn of the spacecraft. In practical applications, consideration of the constrained nature of the angular momentum of a spacecraft is necessary in some cases (especially when the attitude of the spacecraft is controlled using inertial actuators, i.e., control moment gyroscopes,[7] and this constraint becomes significant). This paper is devoted to finding the optimal program for the spatial reorientation of a spacecraft in the minimum time taking into account the constraint imposed on the angular momentum. The optimal control program is synthesized by the classical method.[9]

1. Statement of the problem and the equations of motion

The problem of transferring a spacecraft from an initial oriented position to a position with an assigned orientation in the optimal manner is solved. Spatial reorientation is understood to be a shift of the $OXYZ$ axes associated with the spacecraft body from one known angular position to another known (usually assigned) angular position in a finite time T. In this case the parameters of the turn (for example, the components of the quaternion of the turn) are known *a priori*, back before the beginning of the manoeuvre; the initial angular misalignments can take any value (from a few degrees to 180°). The angular orientation of the $OXYZ$ right-handed rectangular system of coordinates (as well as its initial position $OX_{\text{in}}Y_{\text{in}}Z_{\text{in}}$ and final position $\text{OX}_\text{f}\text{Y}_\text{f}\text{Z}_\text{f}$) is determined relative to the chosen system of coordinates (the reference basis **I**). The most widely encountered case, in which the reference system of coordinates is an inertial system of coordinates, is considered. It is assumed that control of the angular position of the spacecraft is achieved by means of actuators that create moments relative to the three principal central axes of inertia of the spacecraft. The angular motion of a spacecraft as a rigid body is described by the dynamical Euler equations[10]

$$J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 = M_1, \quad J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3 = M_2,$$
$$J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 = M_3 \tag{1.1}$$

where the J_i are the principal central moments of inertia of the spacecraft, M_i are the projections of the principal moments of the external and internal forces onto the principal central axes of inertia of the spacecraft, and ω_i are the projections of the absolute angular velocity vector $\boldsymbol{\omega}$ onto the axes of the attached basis **E**, formed by the principal axis of the ellipsoid of inertia of the spacecraft (i = 1, 2, 3).

We will describe the spatial motion of the spacecraft using the mathematical apparatus of quaternions (Rodrigues–Hamilton parameters). The motion of the attached basis **E** relative to the reference basis **I** will be specified by the quaternion Λ.[1] To be specific, we will assume that the basis **I** is inertial. In this case, the following kinematic equations hold[1]

$$2\dot{\lambda}_0 = -\lambda_1\omega_1 - \lambda_2\omega_2 - \lambda_3\omega_3, \quad 2\dot{\lambda}_1 = \lambda_0\omega_1 + \lambda_2\omega_3 - \lambda_3\omega_2 \quad (1\ 2\ 3) \tag{1.2}$$

In quaternion form, the equivalent equation is $2\dot{\Lambda} = \Lambda \circ \boldsymbol{\omega}$, where λ_j are the components of the quaternion Λ (j = 0, 1, 2, 3) and $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. Here and below (1 2 3) means that two more relations are obtained from the preceding relation by cyclic permutation of the subscripts 1, 2, and 3.

Under the conditions of space flight, a special feature of control is the small magnitude of the perturbing moments caused by the interaction of the spacecraft with external fields and the resistance of the medium. The motion of a spacecraft about its centre of mass is controlled by varying the moment of the external (or internal) forces **M**. It is assumed that the total impulse from the perturbing moments is negligibly small compared with the control impulse. In this case the principal moment of the forces **M** (the variables M_i) is determined mainly by the control moment produced by the system of actuators.

Suppose the magnitude of the angular momentum vector of the spacecraft cannot exceed a certain value H_0 during manoeuvres about the centre of mass, i.e., the condition

$$J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 \le H_0^2 \tag{1.3}$$

where $H_0 > 0$ is a specified positive value, must hold.

The boundary conditions of the control problem (the initial and final states of the spacecraft) have the following form

$$\Lambda(0) = \Lambda_{\text{in}}, \quad \Lambda(T) = \Lambda_{\text{f}} \tag{1.4}$$

where T is the time taken for the spacecraft to complete the reorientation manoeuvre.

In order for the control problem to be closed, we will introduce an optimizable functional. In many cases (including a spacecraft with inertial means of attitude control), an important characteristic is the duration of the turn, and the problem of turning in the minimum time is of interest. The optimality index (the optimizable functional) has the form

$$G = \int_0^T dt \tag{1.5}$$

The problem of the optimal control of a banked turn is stated as follows: it is required to transfor a spacecraft from a state corresponding to the former condition in (1.4) to a state corresponding to the latter condition in (1.4) in accordance with Eqs. (1.1) and (1.2) in the presence of constraint (1.3) with the condition that functional (1.5) should have its minimum value.

When a constraint of the form (1.3) is imposed on the motion of a spacecraft, the control problem stated is fairly important. The results of its solution can be useful to developers of orientation systems for spacecraft equipped with gyroscopic mechanisms, i.e., gyrodynes. In this case, control of a spacecraft turn is achieved by redistributing the angular momentum between the system of gyroscopes and the spacecraft body;[7] the total angular momentum of the spacecraft as a rigid body with rotating masses is equal to or close to zero. The control of a system of gyrodynes in order to produce the programmed motion of a spacecraft by creating the necessary moments M_1, M_2, M_3 is a separate, independent problem (these problems are not considered here). We merely note for an assigned spacecraft turning regime to be realized without having to use other actuators (beside the gyrodynes), for example, rocket thrusters, the total angular momentum of the gyro system must lie within the closed region S (it depends on the design characteristics), which determines the control possibilities of the gyro system, over the entire control interval $[0,T]$. During the development, analysis, perfection, and simulation of algorithms for the attitude control of a spacecraft with control moment gyroscopes, it is assumed that the region S of admissible angular momentum values of the system of control moment gyroscopes is confined to a sphere. This assumption has been used by many researchers;[11–17] it is valid for a large number (if not the majority) of spacecraft (such as the Mir orbital station, the Gamma astrophysical laboratory, the Alpha international space station and others).[13–17]

Since the use of control moment gyroscopes in a turning regime presumes that the total angular momentum of the gyro system would not exceed the admissible value, a constraint, which is formalized for the angular velocity vector, is imposed on the motion of the spacecraft. If the condition $\mathbf{L}+\mathbf{H}\approx 0$, where $\mathbf{L}$ is the angular momentum of the spacecraft body, and $\mathbf{H}$ is the angular momentum of the system of control moment gyroscopes, is taken into account, satisfaction of constraint (1.3) means that the evolution of the vector $\mathbf{H}$ of the gyro system during the spacecraft motion will satisfy the condition that it lies within a region confined by a sphere; therefore, the turn occurs using only the control moment gyroscopes (the vector $\mathbf{H}$ does not extend beyond this region without additional input to the action of the control thrusters).

2. Solution of the problem of optimal control of a spacecraft turn

We will solve the problem using Pontryagin's maximum principle.[9] The functional to be minimized (1.5) and constrain't (1.3) do not contain the components M_i of the moment of the forces in explicit form. Therefore, when we construct the Hamiltonian function, we will take into account only the kinematic equations of motion (1.2), in which the variables ω_i are the required functions to be minimized. We will introduce the conjugate variables ψ_j ($j=0, 1, 2, 3$), which correspond to the quaternion components λ_j. The Hamiltonian function of the problem has the form

$$\Gamma = -1 - \psi_0(\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3)/2 + \psi_1(\lambda_0\omega_1 + \lambda_2\omega_3 - \lambda_3\omega_2)/2 +$$
$$+ \psi_2(\lambda_0\omega_2 + \lambda_3\omega_1 - \lambda_1\omega_3)/2 + \psi_3(\lambda_0\omega_3 + \lambda_1\omega_2 - \lambda_2\omega_1)/2$$

The equations for the conjugate variables ψ_j have the form[9] $\dot\psi_j = -\partial\Gamma/\partial\lambda_j (j = 0, 1, 2, 3)$, or in expanded form

$$\dot\psi_0 = -(\psi_1\omega_1 + \psi_2\omega_2 + \psi_3\omega_3)/2, \quad \dot\psi_1 = (\psi_0\omega_1 + \psi_2\omega_3 - \psi_3\omega_2)/2 \ (1\ 2\ 3) \tag{2.1}$$

After some reduction, we obtain

$$\Gamma = -1 + (\omega_1 p_1 + \omega_2 p_2 + \omega_3 p_3)/2$$

where

$$p_1 = \lambda_0\psi_1 + \lambda_3\psi_2 - \lambda_1\psi_0 - \lambda_2\psi_3 \ (1\ 2\ 3)$$

It follows from Eq. (2.1) that the set of variables ψ_0, ψ_1, ψ_2 and ψ_3 has the properties of quaternions. Henceforth we will assume that the corresponding conjugate variables are components of the quaternion Ψ, for which the equation $2\dot\Psi = \Psi \circ \boldsymbol{\omega}$ holds. Then the vector $\mathbf{p}=\{p_1, p_2, p_3\}$ can be written in the quaternion form $\mathbf{p} = \mathrm{vect}(\tilde\Lambda \circ \Psi)$, if p_1, p_2 and p_3 are the projections of the vector $\mathbf{p}$ onto the axes of the attached basis $\mathbf{E}$. Here vect$(\cdot)$ denotes the operation of isolating the vector part of a quaternion,[1] and $\tilde\Lambda$ is the quaternion conjugate to the quaternion Λ. The Hamiltonian function takes the form $\Gamma=-1+\boldsymbol{\omega}\,\mathbf{p}/2$.

We will investigate the properties of the solution of the conjugate system of Eq. (2.1). The systems of differential Eqs. (1.2) and (2.1) are similar with respect to the coefficients ω_1, ω_2 and ω_3. The solutions of kinematic Eq. (1.2) for the variables λ_j and of Eq. (2.1) for the variables ψ_j differ with respect to their initial conditions, and the quaternions Ψ and Λ are related to one another as follows:

$$\Psi = C_E \circ \Lambda; \quad 2\dot\Psi = 2C_E \circ \dot\Lambda = C_E \circ \Lambda \circ \boldsymbol{\omega} = \Psi \circ \boldsymbol{\omega}$$

where C_E = const is a constant quaternion. It is still unknown and will be determined when solving the optimal control problem (after solving the maximum principle boundary-value problem). A necessary and sufficient condition for the system of equations consisting of Eqs. (1.2) and (2.1) to be non-degenerate is vect $C_E \neq 0$. In the opposite case of ψ_0: $\lambda_0=\psi_1$: $\lambda_1=\psi_2$: $\lambda_2=\psi_3$: λ_3, Eqs. (1.2) and (2.1) will not be independent (in the sense that only four of the eight equations will be independent), $\mathbf{p}=0$, and the solution of the problem will be meaningless. Therefore, to determine the optimal control, the quaternions Λ and Ψ are assumed not to be identical (the functions ψ_j are not proportional to the variables λ_j), and only the version $\mathbf{p} \neq 0$ is considered.

Differentiating the expressions for p_i ($i=1, 2, 3$) and substituting the expressions for λ_j and ψ_j ($j=0, 1, 2, 3$) into them, we obtain the following system of equations for the time functions p_i

$$\dot p_1 = \omega_3 p_2 - \omega_2 p_3 \ (1\ 2\ 3) \tag{2.2}$$

or, in vector form,

$$\dot{\mathbf{p}} = -\omega \times \mathbf{p} \tag{2.3}$$

Eq. (2.3) reflects the rotation of the vector $\mathbf{p}$ with angular velocity $-\boldsymbol{\omega}$ about the attached basis $\mathbf{E}$. In turn, the attached basis $\mathbf{E}$ performs angular motion about the reference basis $\mathbf{I}$ with angular velocity $\boldsymbol{\omega}$. As a result, the vector $\mathbf{p}$ is fixed in the reference system. By virtue of the fact that $|\mathbf{p}|=\mathrm{const}$, we will assume below that the vector $\mathbf{p}$ is normalized: $|\mathbf{p}|=1$.

Thus, the problem of determining the optimal control reduces to solving system of Eqs. (1.1), (1.2) and (2.2) under the condition that the control itself is selected on the basis of the requirement for maximizing the Hamiltonian function. The boundary conditions with respect to the angular positions Λ_{in} and Λ_{f} specify a family of solutions of $\mathbf{p}(t)$, which has the form[1] $\mathbf{p} = \tilde\Lambda \circ \mathbf{c}_E \circ \Lambda$, where $\mathbf{c}_E = \Lambda_{\mathrm{in}} \circ \mathbf{p}(0) \circ \tilde\Lambda_{\mathrm{in}} = \mathrm{const}$.

According to its physical meaning, $\mathbf{c}_E = \operatorname{vect} C_E = \operatorname{vect}(\Psi \circ \tilde{\Lambda})$, while the components of the vector $\mathbf{c}_E$ are projections of the vector $\mathbf{p}$ onto the axes of the inertial system of coordinates.

The direction of the vector $\mathbf{c}_E$ depends on the initial and final angular positions of the spacecraft. In order for the spacecraft to have the required orientation at the right-hand end $\Lambda(T)=\Lambda_f$, the vector $\mathbf{c}_E$ (or the value of the vector $\mathbf{p}$ at the initial point in time) must be determined from the solutions of system (1.2). The problem of finding the optimal control consists of investigating the dynamical Euler Eq. (1.1), the kinematic equations of motion (1.2) and the conjugate Eq. (2.2) for the motion of the vector $\mathbf{p}$ indicated. Differential Eq. (2.2), together with the requirement for maximizing the Hamiltonian function Γ, are necessary conditions of optimality. The value of $\Psi(0)$ (and, accordingly, $\mathbf{p}(0)$) is selected such that the maximum principle boundary-value problem has a solution. The coupling equations are expressed by system of Eq. (1.2) with simultaneous satisfaction of constraint (1.3) on the motion of the spacecraft. The boundary conditions Λ_{in} and Λ_{f} and the conditions for a maximum of the function Γ determine the optimum solution for $\boldsymbol{\omega}(t)$, $\Lambda(t)$ and $\mathbf{p}(t)$.

We will find the necessary conditions of optimality in the form of a functional dependence of the control variables on the phase coordinates and conjugate variables. The control functions are the projections ω_i of the angular velocity vector $\boldsymbol{\omega}$ onto the axis of the attached basis $\mathbf{E}$. The necessary condition of optimality has the form

$$\Gamma = (p_1\omega_1 + p_2\omega_2 + p_3\omega_3)/2 - 1 \to \max$$

To obtain equations that define the optimal solution, we make a replacement of variables. We introduce the notation $L_i = J_i\mu_i$ and $\mu_i = p_i/J_i$ ($i = 1, 2, 3$). We then have

$$\Gamma = (L_1\mu_1 + L_2\mu_2 + L_3\mu_3)/2 - 1$$

To maximize the function Γ, the requirement $|\boldsymbol{\omega}| \to \max$ must be satisfied at each point in time (the angle between the vectors $\boldsymbol{\omega}$ and $\mathbf{p}$ is acute) with simultaneous satisfaction of condition (1.3) (i.e., the condition that the vector $\boldsymbol{\omega}$ is in the region of admissible values of Ω). Therefore, the equality

$$J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 = H_0^2 \tag{2.4}$$

holds when the rotation of the spacecraft is optimal. The function Γ clearly takes a maximum value under the condition $L_1^2 + L_2^2 + L_3^2 = H_0^2$, when the vectors $\mathbf{L} = \{L_1, L_2, L_3\}$ and $\boldsymbol{\mu} = \{\mu_1, \mu_2, \mu_3\}$ have the same direction. The relations

$$L_i = \frac{H_0 p_i}{J_i\sqrt{p_1^2/J_1^2 + p_2^2/J_2^2 + p_3^2/J_3^2}} \tag{2.5}$$

then hold. Hence we have $p_i = qj_i^2\omega_i$, where $q > 0$ is a scalar quantity. It can be shown that $q = \text{const}$.

For this purpose, we test the equality $p_1^2/j_1^2 + p_2^2/j_2^2 + p_3^2/j_3^2 = \text{const}$. We take the derivative of the left-hand side of this equality with respect to time and replace the derivatives of the components p_i of the vector $\mathbf{p}$ according to relations (2.2); in the expression obtained the components ω_i of the vector $\boldsymbol{\omega}$ are replaced according to the formulae presented above that relate the p_i and ω_i (or L_i). In fact,

$$\begin{aligned} p_1\dot{p}_1/J_1^2 + p_2\dot{p}_2/J_2^2 + p_3\dot{p}_3/J_3^2 &= \omega_1 p_2 p_3/J_3^2 - \omega_1 p_2 p_3/J_2^2 + \omega_1 p_2 p_3/J_2^2 - \\ &- \omega_2 p_1 p_3/J_3^2 + \omega_2 p_1 p_3/J_3^2 - \omega_1 p_2 p_3/J_3^2 \equiv 0 \end{aligned}$$

from which the validity of the statement regarding the constancy of the coefficient q follows.

The optimal motion of the spacecraft is completely specified by the system consisting of differential Eq. (2.2) and the equations

$$\omega_i = \frac{H_0 p_i}{J_i^2\sqrt{p_1^2/J_1^2 + p_2^2/J_2^2 + p_3^2/J_3^2}}, \quad i = 1, 2, 3 \tag{2.6}$$

when the initial condition

$$\Lambda(0) = \Lambda_{\text{in}} \tag{2.7}$$

and the boundary condition

$$\Lambda(T) = \Lambda_{\text{f}} \tag{2.8}$$

are ensured for the solution $\Lambda(t)$ of system (1.2).

The calculated values of the control moments M_i can be determined from the condition that the spacecraft moves along an assigned kinematic trajectory by solving the inverse dynamical problem. Substituting expressions (2.6) into system of Eq. (2.2), we obtain

$$J_i^2\dot{\omega}_i = \omega_{i-1}\omega_{i+1}(J_{i+1}^2 - J_{i-1}^2), \quad i = 1, 2, 3$$

or, in vector form, ($j\dot{\boldsymbol{\omega}} = -j^{-1}(\boldsymbol{\omega} \times (j^2\boldsymbol{\omega}))$), whence we have

$$\mathbf{M} = \boldsymbol{\omega} \times (J\boldsymbol{\omega}) - J^{-1}(\boldsymbol{\omega} \times (J^2\boldsymbol{\omega})) \tag{2.9}$$

where $J = \operatorname{diag}(J_1, J_2, J_3)$ is the inertia tensor of the spacecraft, and the vectors $\mathbf{p}$ and $\boldsymbol{\omega}$ are the solution of system of Eqs. (2.2) and (2.6).

The optimal turn of a spacecraft in the minimum time occurs with the maximum admissible angular momentum. The problem of synthesizing the optimal control reduces to finding the law of variation of the vector $\mathbf{p}(t)$ for which boundary condition (2.8) is satisfied as a result of the motion of the spacecraft according to system of Eqs. (1.2), (2.2) and (2.6) with initial condition (2.7).

The main problem involves finding a value of the vector $\mathbf{p}(0)$ such that equality (2.8) is satisfied as a result of the motion of the spacecraft in accordance with Eqs. (1.1), (1.2), (2.2) and (2.6). It is practically impossible to construct the general solution of this system of equations. The difficulty lies in determining the boundary conditions $\mathbf{p}(0)$ and $\mathbf{p}(T)$, which are related by the expression

$$\mathbf{p}(T) = \tilde{\Lambda}_f \circ \Lambda_{in} \circ \mathbf{p}_0 \circ \tilde{\Lambda}_{in} \circ \Lambda_f = \tilde{\Lambda}_t \circ \mathbf{p}(0) \circ \Lambda_t$$

where $\Lambda_t = \tilde{\Lambda}_{in} \circ \Lambda_f$ is the quaternion of the turn.

System of Eqs. (1.2), (2.2) and (2.6) has an analytical solution in elementary functions only for dynamically symmetrical and dynamically spherical bodies. A similarly formulated optimal control problem for a spherical body was examined in detail in Ref. 1. For a dynamically symmetrical body with $J_1 \neq J_2 = J_3$, the solution $\mathbf{p}(t)$ can be found in analytical form. The optimal function $\mathbf{p}(t)$ is written in the following manner

$$p_1 = p_{10}, \quad p_2 = p_{20}\cos\sigma + p_{30}\sin\sigma, \quad p_3 = -p_{20}\sin\sigma + p_{30}\cos\sigma \tag{2.10}$$

where

$$\mathbf{p}_0 = \mathbf{p}(0), \quad \sigma = \dot{\alpha}t + \delta_0, \quad \dot{\alpha} = \text{const} = \omega_1 - \sqrt{\omega_2^2 + \omega_3^2}\,p_{10}/\sqrt{1 - p_{10}^2}$$

(α is the intrinsic rotation rate).

The specific value $\mathbf{p}_0$ is determined exclusively so that as a result of rotation of the spacecraft according to Eqs. (2.2) and (2.6) with the initial conditions

$$\mathbf{p}(0) = \mathbf{p}_0 \left(\omega_i(0) = \frac{H_0 p_{i0}}{J_i^2 \sqrt{p_{10}^2/J_1^2 + p_{20}^2/J_2^2 + p_{30}^2/J_3^2}} \right)$$

the solution of Eqs. (1.2) with initial conditions (2.7) satisfies equality (2.8). The solution of system of Eqs. (2.2) and (2.6) is found in the form of regular precession[10] (conical precessing motion). The programmed values of the projections of the vector of the angular velocity $\omega(t)$ onto the attached axes will be the following

$$\begin{aligned} \omega_1 &= \omega_{10} = \dot{\alpha} + \dot{\beta}\cos\vartheta \\ \omega_2 &= \dot{\beta}p_2 = \dot{\beta}\sin\vartheta\sin(\dot{\alpha}t + \sigma_0) \\ \omega_3 &= \dot{\beta}p_3 = \dot{\beta}\sin\vartheta\cos(\dot{\alpha}t + \sigma_0) \end{aligned} \tag{2.11}$$

where ϑ is the angle between the longitudinal axis of the spacecraft and the vector $\mathbf{p}$, $\dot{\alpha}$ is the angular intrinsic rotation rate (about the longitudinal axis), and $\dot{\beta}$ is the angular precession rate (about the vector $\mathbf{p}$).

The optimality conditions for the parameters ϑ, $\dot{\alpha}$ and $\dot{\beta}$ take the form

$$J_1^2(\dot{\alpha} + \dot{\beta}\cos\vartheta)^2 + J_2^2\dot{\beta}^2\sin^2\vartheta = H_0^2, \quad \dot{\alpha}\cos\vartheta + \dot{\beta} \to \max$$

In the case of a spacecraft with dynamical symmetry ($J_2 = J_3$), relations (2.10) together with equalities (2.11) form a solution of system of Eqs. (1.2), (2.2) and (2.6) under condition (2.4). The vector $\mathbf{p}$ describes a cone about the longitudinal OX axis in the attached system of coordinates. Under such a control, an axisymmetrical body moves along a "conical trajectory" (Ref. 10). The spacecraft is transferred from the angular position Λ_{in} to the angular position Λ_f by simultaneously rotating it about the vector $\mathbf{c}_E$, which is fixed relative to the inertial basis $\mathbf{I}$, through the angle β and about its longitudinal axis through the angle α. Using the mathematical apparatus of quaternions to describe the rotations of a rigid body about its centre of mass, we obtain the relation

$$\Lambda_f = \Lambda_{in} \circ \exp(\mathbf{p}_0\beta/2) \circ \exp(\mathbf{e}_1\alpha/2)$$

where $\mathbf{e}_1$ is the unit vector along the longitudinal axis of the spacecraft.

The dependence of the parameters $\mathbf{b}_0$, α and β on the limiting angular positions Λ_{in} and Λ_f is specified by the system of equations

$$\begin{gathered} \cos\frac{\beta}{2}\cos\frac{\alpha}{2} - p_{10}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = \nu_0, \quad \cos\frac{\beta}{2}\sin\frac{\alpha}{2} + p_{10}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = \nu_1 \\ p_{20}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} + p_{30}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = \nu_2, \quad -p_{20}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} + p_{30}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = \nu_3 \end{gathered} \tag{2.12}$$

where ν_0, ν_1, ν_2 and ν_3 are the components of the quaternion of the turn $\Lambda_t = \tilde{\Lambda}_{in} \circ \Lambda_f$, α is the angle of rotation of the spacecraft about the longitudinal axis, and β is the angle of rotation about the vector $\mathbf{p}$. Here it is assumed that $|\alpha| \leq \pi$ and $0 \leq \beta \leq \pi$.

In essence, the optimization was reduced to determining the characteristic ϑ (or $p_{10} = \cos\vartheta$), while the angles of rotation α and β are calculated uniquely from system (2.12). The optimality conditions will be obeyed if it is required that

$$J_1^2(\alpha + \beta\cos\vartheta)^2 + J_2^2\beta^2\sin^2\vartheta \to \min$$

The optimum values of the angles α, β and ϑ that satisfy the assigned limiting values Λ_{in} and Λ_f in accordance with equalities (2.12) can be determined using the previously proposed system.[18] The turn time is estimated by the quantity

$$T = \sqrt{J_1^2(\alpha + \beta\cos\vartheta)^2 + J_2^2\beta^2\sin^2\vartheta}/H_0$$

The calculated angular precession and intrinsic rotation rates are:

$$\dot{\alpha} = \alpha/T, \quad \dot{\beta} = \beta/T$$

Thus, the kinematic problem of the reorientation of a spacecraft (as a rigid body) has been completely solved. Optimal control of the angular position of a spacecraft is realized using the previously proposed method.[19]

For an arbitrary spacecraft ($J_1 \neq J_2 \neq J_3$), the solution of system of Eqs. (1.2), (2.2) and (2.6) is found by numerical methods (for example, by the method of successive approximations). The vector $\mathbf{p}_0$ is determined by solving the boundary-value problem with conditions (2.7) and (2.8), taking with account relations (1.2), (2.2) and (2.6), which are imposed on the motion.

The quaternions Λ_{in} and Λ_{f}, which assign the orientation of the axes attached to the spacecraft at the initial and final times have arbitrary pre-assigned values. Of course, at the times $t = 0$ and $t = T$ the angular velocities for the nominal spacecraft rotation program specified by Eqs. (2.6) are not equal to zero. Therefore, the following transitional phases are unavoidable: start up, i.e., a transition from a state of rest (in which $\boldsymbol{\omega} = 0$) to a rotation regime with an angular momentum that has the maximum value H_0, and stopping, i.e., reduction of the angular momentum of the spacecraft to zero. If the initial (Λ_{in}) and final (Λ_{f}) angular positions and the value of H_0 are such that the start up and stopping times are negligibly small (compared to the time T of the entire turn), the imparting of the necessary angular momentum H_0 to the spacecraft and the reduction of the existing angular momentum to zero can be considered to be almost instantaneous. In this case, the phase between start up and stopping, during which condition (1.3) changes to a strict equality, is the main phase. The necessary conditions of optimality for the phase of rotation of the spacecraft with a constant value of the angular momentum take the form of (2.2) and (2.6). The control moments M_i needed to maintain the optimum motion regime are found from the dynamical Euler Eq. (1.1) (for known forms of the functions $\omega_i(t)$). The value of the vector $\mathbf{p}$ at the time $t = 0$ is a decisive factor for finding the optimal solutions $\mathbf{p}(t)$ and $\boldsymbol{\omega}(t)$. After solving the kinematic problem of turning a spacecraft from the angular position $\Lambda(0) = \Lambda_{\text{in}}$ to the angular position Λ_{f}, taking Eqs. (1.2) and (2.2) into account, in which relation (2.6) holds, we find the calculated value of the vector $\mathbf{p}_0$ and the vector $\mathbf{c}_E = \text{const}$ that corresponds to it. In the phase of rotation of the spacecraft with a constant magnitude of the angular momentum specified by condition (2.4), the mathematical expression for the moment $\mathbf{M}$ takes the form (2.9) (due to the existence of Eqs. (2.2) and equalities (2.6), which are obeyed by the functions $\omega_{\text{i}}(t)$).

In numerous cases, the actuators of a spacecraft orientation system are control moment gyroscopes (CMGs). Their use in a turning regime requires that the total angular momentum $\mathbf{H}$ of the gyro system does not exceed the admissible value. Control by moment gyroscopes (by a system of gyrodynes) that create a moment $\mathbf{M}$ for producing the required programmed motion of the spacecraft about the centre of mass (in accordance with Eqs. (1.1)) is a separate independent problem. These questions, as well as questions concerning the accuracy of the determination of the optimal moment $\mathbf{M}$, allowing for the dynamics of the control gyroscopes, are not considered here. In general, the region S of possible values of the angular momentum of a system of CMGs, intended for attitude control of a spacecraft, is confined to a sphere of radius H_{ad} with its centre at the origin of the attached system of coordinates $Oxyz$, where $H_{\text{ad}} > 0$ is the maximum admissible magnitude (absolute value) of the total angular momentum of the system of CMG actuators. "Saturation" of the system of CMGs, i.e., achievement of the boundary of the region S of possible values by the angular momentum of the CMGs, sets in at the time when the equality $|\mathbf{H}| = H_{\text{ad}}$ is satisfied. Further attitude control using some CMGs is impossible, and "unloading" of the CMG system,[7] i.e., the elimination of the angular momentum of the CMGs by applying a moment of forces of a different nature, such as magnetic forces,[2] including orientation thrusters, etc. For small Gamma spacecraft a control for performing turns is constructed only using CMGs in the overwhelming majority of cases.[14] In order that a spacecraft turn manoeuvre should occur over the entire time interval $0 \le t \le T$ without the need for "unloading" the CMGs, satisfaction of the condition $|\mathbf{L}| \le H_0$ is required, where H_0 is an assigned constant quantity that satisfies the inequality $0 < H_0 < H_{\text{ad}}$. Such spacecraft motions are considered to be admissible (in the sense of controlling the attitude of the spacecraft without "unloading" the CMGs). The difference $H_{\text{ad}} - H_0$ (storage of angular momentum of a certain kind) is necessary for the guaranteed absence of the angular momentum of the CMG system outside the region S of possible values even under the action of various perturbing moments. Hence, the presence of constraint (1.3) in the statement of the optimal control problem, as well as the applied value of the solution obtained for use in spacecraft with an orientation system based on control moment gyroscopes, become understandable.

3. Results of mathematical modelling

The purpose of the mathematical syntheses presented above is to obtain an answer to the question of what should the motion of a spacecraft about its centre of mass be in order for, first, rotation of the spacecraft from the known angular position Λ_{in} to bring the spacecraft into the assigned final angular position Λ_{f}, second, for constraint (1.3) to be satisfied during the manoeuvre, and, third, for the time T needed to achieve the required attitude Λ_{f} to be a minimum. For a dynamically symmetrical spacecraft, the problem of an optimal turn is completely solved in formulation (1.1)–(1.5). We now present a numerical solution of the problem of the optimal control of a programmed turn of a spacecraft in the minimum time with a constrained magnitude of the angular momentum. As an example, we will examine the banked turn of a certain spacecraft with the following moments of inertia

$$J_1 = 8.82 \times 10^3\ \text{kg m}^2, J_2 = 3.16 \times 10^4\ \text{kg m}^2, J_3 = 3.00 \times 10^4\ \text{kg m}^2$$

From the initial angular position Λ_{in}, which is identical to the axes of the reference basis $\mathbf{I}$, to the assigned final angular position $\Lambda_{\text{f}} = \Lambda_{\text{pr}}$. The values of the elements of the quaternion Λ_{pr} that assign the required angular position of the spacecraft after the turn were as follows:

$$\lambda_0 = 0.259,\ \lambda_1 = 0.683,\ \lambda_2 = 0.592,\ \lambda_3 = 0.342$$

The turn quaternion Λ_t adopted corresponds to a version in which the final rotation vector[2] (the Euler axis) makes the same angle with the OX axis and with the plane perpendicular to this axis, i.e., reflects the apparently most difficult case of reorientation of a rigid body that does not have spherical symmetry.

The maximum principle boundary-value problem, i.e., the determination of the vector $\mathbf{p}_0$ for optimal motion, can be solved by the method of successive approximations. The solution of the analogous optimal control problem for a dynamically symmetrical body with moment of inertia J_1 about the longitudinal axis and moment of inertia $J_{tr} = (J_2 + J_3)/2$ about the transverse axis is taken as the initial approximation. The solution for the initial approximation $\mathbf{p}_0^{(0)}$ is found from system of Eq. (2.12) with minimization of the quantity $J_1^2(\alpha+\beta p_{10})^2 + J_{tr}^2\beta^2(1-p_{10}^2)$. The time-optimal motion has the form of a precession of the rigid body about an axis, fixed in the inertial system of coordinates. Therefore,

$$\mathbf{L}^2 = J_1^2(\dot{\alpha}+\dot{\beta}p_{10})^2 + J_{tr}^2\dot{\beta}^2(1-p_{10}^2) \text{ и } \mathbf{L}^2 = H_0^2.$$

We recall that for a first approximation $\dot{\alpha} = \alpha/T$, and $\dot{\beta} = \beta/T$. Multiplying the left- and right-hand sides by T^2, we obtain

$$H_0^2T^2 = J_1^2(\alpha+\beta p_{10})^2 + J_{tr}^2\beta^2(1-p_{10}^2)$$

We start from the fact that $H_0 = \text{const}$. The condition $T \to \min$ will hold if we require

$$J_1^2(\alpha+\beta p_{10})^2 + J_{tr}^2\beta^2(1-p_{10}^2) \to \min$$

Next, the vector $\mathbf{p}_0$ is refined by modelling the motion in accordance with Eqs. (1.2), (2.2) and (2.6). First, system of Eqs. (1.2), (2.2) and (2.6) is integrated with the initial conditions

$$\mathbf{p}(0) = \mathbf{p}_0, \quad \omega_i(0) = H_0p_{i0}/(J_i^2\sqrt{p_{10}^2/J_1^2 + p_{20}^2/J_2^2 + p_{30}^2/J_3^2}), \quad \Lambda(0) = \Lambda_{in} \tag{3.1}$$

The estimate $\varepsilon = \sum_{j=0}^{3}\lambda_j(t)\lambda_{jf}$ is calculated at each integration step. At the time T, when the value of ε is a maximum, the parameters $\Lambda(T)$ are recorded and stored: $\Lambda_{mod} = \Lambda(T)$, and Λ_{mod} is the expected (predicted) angular position to which the spacecraft transfers if the value of the vector $\mathbf{p}_0$ is not altered. The aim of the process of approximating the vector $\mathbf{p}_0$ to the solution sought is to reduce the discrepancies $\lambda_{jf} - \lambda_{jmod}$ to zero. To correct the vector $\mathbf{p}_0$, we introduce the function

$$F = \sum_{j=0}^{3}(\lambda_{jf} - \lambda_{j\text{mod}})^2$$

The optimum (sought) value of the vector $\mathbf{p}_0$ corresponds to the minimum of the function F. The objective function F is minimized with respect to the argument $\mathbf{p}_0$ by the gradient method[20] (or the method of steepest descents). Taking into account that $|\mathbf{p}_0| = 1$, we can conveniently change to the coordinates $\vartheta = \arccos p_{10}$ and $\varphi = \arctan(p_{20}/p_{30})$. In spherical coordinates

$$p_{10} = \cos\vartheta, \quad p_{20} = \sin\vartheta\sin\varphi, \quad p_{30} = \sin\vartheta\cos\varphi \tag{3.2}$$

By varying the values of the variables ϑ and φ, we obtain the variation of the function F (in terms of the solution of system of Eqs. (1.2), (2.2) and (2.6), in which the initial conditions are specified by expressions (3.2) and (2.6), which relate $\omega_i(0)$ and p_{i0}, and the definition of Λ_{mod}). The system of Eqs. (1.2), (2.2) and (2.6) is intergrated and Λ_{mod} is determined at each approximation step and for each correction of the parameters ϑ and φ (and, accordingly, of the vector $\mathbf{p}_0$). For the kth approximation of $\vartheta^{(k)}$, $\varphi^{(k)}$ (and thus $\mathbf{p}_0{}^{(k)}$), we find the modelled angular position $\Lambda_{mod}^{(k)}$ and the value of the minimized function F. The iteration process is stopped when $F < F_{ad}$, where F_{ad} is a positive value (the tolerance) close to zero. Note that the value of F_{ad} determines the accuracy of the solution of the boundary-value problem $\Lambda(0) = \Lambda_{in}$, $\Lambda(T) = \Lambda_f$ with respect to the angular position at the right-hand end. In fact, the turn accuracy $\Delta\psi$ is related to the function F by the expression

$$F = 2\left(1 - \cos\frac{\Delta\psi}{2}\right)$$

whence we have $\Delta\psi < 4\arcsin(\sqrt{F_{ad}}/2)$. We choose the acceptable bound F_{ad} in accordance with this inequality.

An analysis of system of Eq. (2.12) shows that the component p_{10} cannot exceed the range $|p_{10}| \le \xi$, where $\xi = \sqrt{\nu_0^2 + \nu_1^2}$. The first two equations in system (2.12) lead to the equality

$$\cos^2\frac{\beta}{2} + p_{10}^2\sin^2\frac{\beta}{2} = \xi^2$$

from which we obtain an expression for p_{10}, where $\max |p_{10}| = \xi$.

The search strategy used to find the minimum of the function $j_1^2(\alpha+\beta p_{10})^2 + J_{tr}^2\beta^2(1-p_{10}^2)$ in the interval $[-\xi, \xi]$ can be different (Fibonacci's method,[20] the method of dividing the interval in half etc.). The golden section method[20] turns out to be the best. After specifying the solution accuracy Δp_{10} equal to 10^{-7}, we obtain the required value $\mathbf{p}_0^{(0)}$ already at the nth iteration, where $n \ge (7 + 1\text{g}\xi)/(1\text{g}(\sqrt{5}+1) - 1\text{g}2) + 1$. The minimum is usually found at the point for which $\text{sign}\, p_{10} = \text{sign}(\nu_0\nu_1)$.

The function $F(\vartheta, \varphi)$ is a convex function (in Vasil'ev's terminology).[20] Using the notation $u = (\vartheta, \varphi)^T$ for the vector of the argument and $\text{grad}\, F = (F_\vartheta, F_\varphi)^T$ for the gradient of the function $F(\vartheta, \varphi)$, we write the fundamental rule of the gradient method: $u^{(k+1)} = u^{(k)} - \rho\, \text{grad}\, F$, where $\rho > 0$. Using the property $\min F(\vartheta, \varphi) = 0$, we adopt the following scheme for generating the sequence of approximations $(\vartheta^{(k)}, \varphi^{(k)})$ (the minimizing sequence)[20]

$$u^{(k+1)} = u^{(k)} - F(u^{(k)})\text{grad}F/\text{grad}^2F$$

or in expanded form

$$\vartheta^{(k+1)} = \vartheta^{(k)} - F(\vartheta^{(k)}, \varphi^{(k)})[F_\vartheta^2 + F_\varphi^2]^{-1} F_\vartheta$$

$$\varphi^{(k+1)} = \varphi^{(k)} - F(\vartheta^{(k)}, \varphi^{(k)})[F_\vartheta^2 + F_\varphi^2]^{-1} F_\varphi$$

The value of ρ was selected using the condition $\Delta\vartheta F_\vartheta + \Delta\vartheta F_\varphi = -F$, and it was, therefore, assumed that $\rho = F(\vartheta, \varphi)/\text{grad}^2 F$.

In determining the partial derivatives F_ϑ and F_φ (for calculating the gradient of $F(\vartheta, \varphi)$), the increments of the independent variables ϑ and φ were assumed to be of the order of $5 \times 10^{-4} \ldots 10^{-3}$. The accuracy threshold F_{ad} was set at a level of 2×10^{-7}, which corresponds to an attitude error $\Delta\psi$ no greater than 3 min. The convergence of the approximation process to a minimum of the function F is due to the proximity of the initial approximation $\mathbf{p}_0^{(0)}$ to the true extremum of F. Even for the case of a 180-degree turn, the deviation $\Delta p = |\mathbf{p}_0^{(0)} - \mathbf{p}_0^*|$

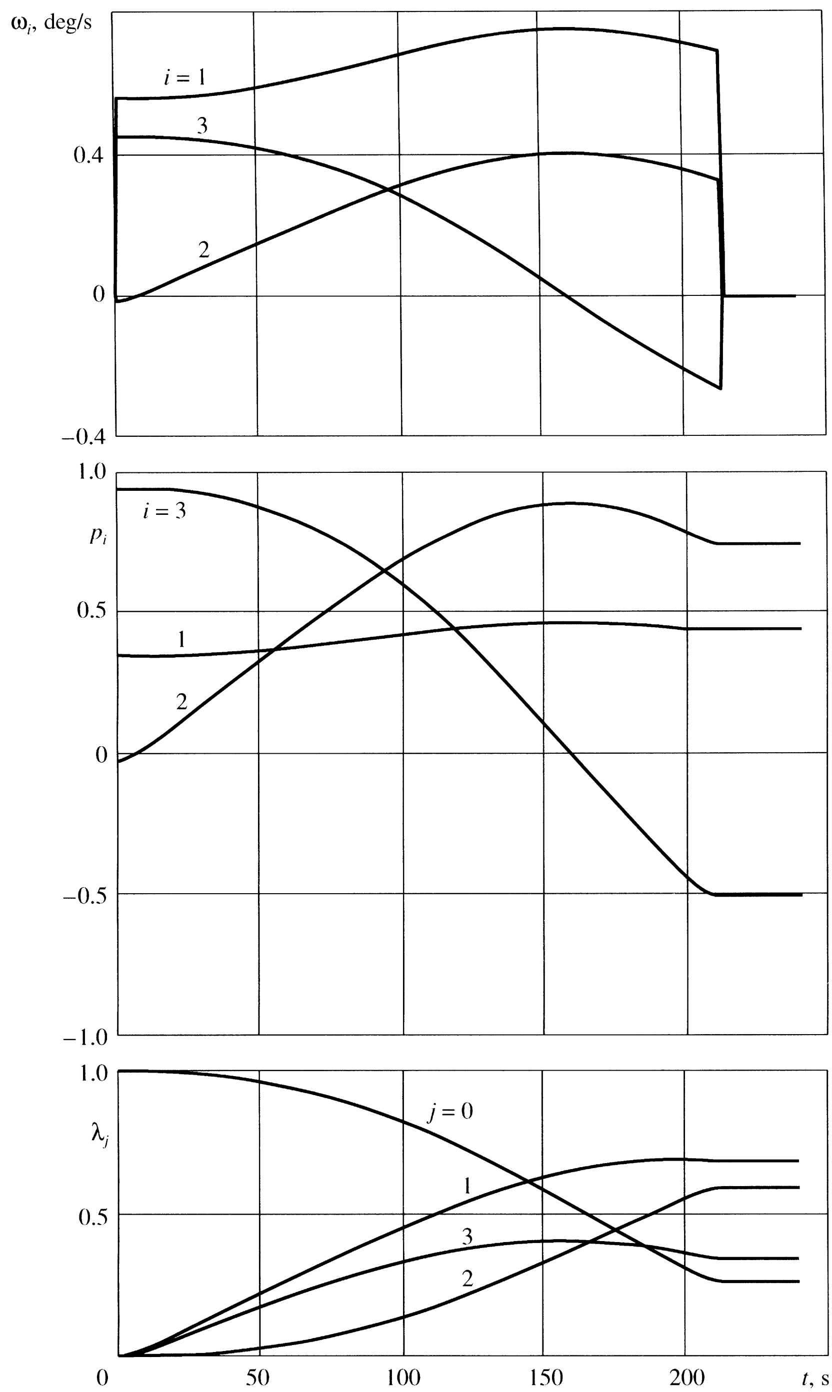

Fig. 1.

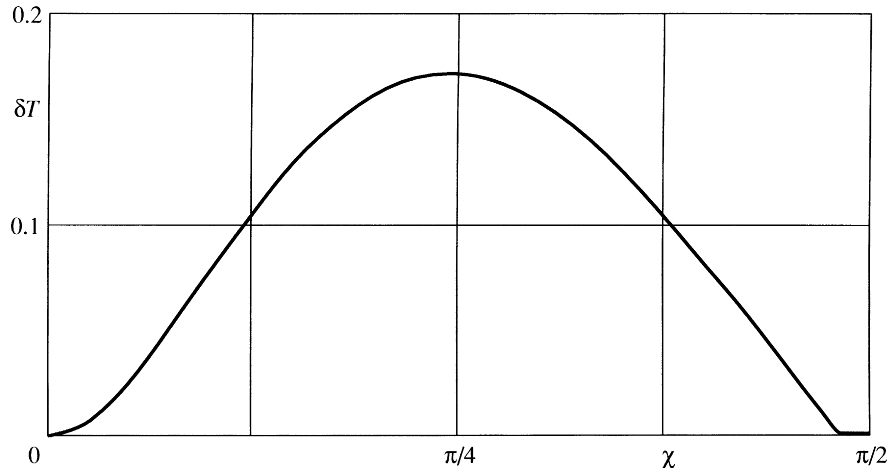

Fig. 2.

does not exceed 2.20×10^{-2}, where $\mathbf{p}_0^{(0)}$ is the initial approximation and $\mathbf{p}_0^*$ is the true value of $\mathbf{p}_0$ at which $F=0$ (i.e., $F(\mathbf{p}_0^*)=0$), and $F^{(0)}=6.47\times10^{-3}$.

Thus, the procedure for calculating the required vector $\mathbf{p}_0$, which specifies the initial data $\mathbf{p}(0)$ and $\boldsymbol{\omega}(0)$ of the optimum motion (the initial data $\Lambda(0)$ do not vary; they are known and equal to Λ_{in}), consists of two successive steps, viz., finding the initial approximation $\mathbf{p}_0^{(0)}$ by the procedure for finding the minimum of the function of the single variable p_{10} (the angles α and β and the value of the minimized function $J_1^2(\alpha+\beta p_{10})^2+J_{\text{tr}}^2\beta^2(1-p_{10}^2)$ are found from the solution of system (2.12) when the value of p_{10} is known), and refinement of the vector $\mathbf{p}_0$ by minimizing the discrepancy function F as a function of the two variables, ϑ and φ, by one of the known numerical methods.[20] The value of the discrepancy function F for a specific value of $\mathbf{p}_0$ is determined by integrating system of Eqs. (1.2), (2.2) and (2.6) with the initial conditions (3.1) (by modelling the motion of the spacecraft) and obtaining the predicted angular position Λ_{mod}.

As a result of solving the kinematic orientation problem of transferring an asymmetrical spacecraft from the angular position $\Lambda(0)=\Lambda_{\text{in}}$ to the angular position $\Lambda(T)=\Lambda_{\text{f}}$ (the optimum turn problem in a momentum formulation), we obtained the calculated value of the vector $\mathbf{p}_0=\{0.344, -0.030, 0.939\}$ and the equality $\mathbf{c}_E=\mathbf{p}_0$ (since $\Lambda_{\text{in}}=\{1, 0, 0, 0\}$). The maximum admissible value of the angular momentum of the spacecraft was assumed to be equal to $H_0=250$ N m s. The required accuracy is achieved at the fifth step of approximating the vector $\mathbf{p}_0$ to the solution required.

The results of the mathematical modelling of the dynamics of the motion of a spacecraft under line-optimal control are presented in Fig. 1. The upper part of this figure shows graphs of the variation of the angular velocities in the system of coordinates $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ attached to the spacecraft in time. The entire turn is completed in a time $T=214$ s. As a result, the spacecraft was turned by 150°. The middle part of Fig. 1 presents the dynamics of the variation of the conjugate variables $p_1(t)$, $p_2(t)$ and $p_3(t)$. Finally, the lower part of Fig. 1 shows graphs of the variation of the components of the quaternion $\Lambda(t)$, which specifies the current attitude of the spacecraft during the rotation manoeuvre: $\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$. The variables λ_j and p_i are smooth functions of time. The far smaller variation of the projection p_1 compared to the variation of p_2 and p_3 is characteristic. The angular velocity component ω_1 also varies less than ω_2 and ω_3 do in the nominal rotation phase (in the time interval in which the angular momentum has a constant value). This confirms that the OX axis is the longitudinal axis. The following rule is observed for the functions $\omega_1(t)$ and $p_1(t)$: for any combinations of the limiting values of Λ_{in} and Λ_{f}, these functions are always sign-invariant and of the same sign.

The size of the decrease in the turn time T compared with the corresponding size for known spacecraft attitude control methods is of interest. The investigation was performed by mathematically modelling of a large number of turns. The efficiency of the control algorithm developed is estimated by the relative reduction of the turn time

$$\delta T = \Delta T/T = (T_{\text{Eul}} - T_{\text{opt}})/T_{\text{Eul}}$$

where T_{opt} is the turn time in the case of optimal control and T_{Eul} is the turn time in the case of rotation of the spacecraft about the Euler axis.

The results of the numerical experiments are presented in Fig. 2 in the form of a graph of δT against the turning conditions (the angle χ between the final rotation vector and the longitudinal axis of the spacecraft). It can be seen that the time saving is a maximum in the vicinity of the point $\chi=\pi/4$.

4. Conclusion

The maximum principle, which is the most thoroughly developed and effective method for solving optimal control problems with constrained control variables, has enabled us to find the necessary conditions of optimality for a spacecraft reorientation regime. The spatial motion of a spacecraft about its centre of mass is described by quaternion variables, which greatly simplify the calculation procedures and significantly reduce the computational cost of the control algorithm, making it more convenient for vehicle-borne implementation. The optimality conditions have been written in analytical form (in the form of a system of equations). It has been shown that the optimal solution belongs to a class of regular motions that are close to the precession of a rigid body about a certain axis fixed in inertial space. In the general case (a turn from a state of rest to a state of rest), a spacecraft reorientation manoeuvre can be divided into three characteristic phases: start up (almost instantaneous imparting of angular velocity) to the maximum admissible angular momentum, rotation with the maximum magnitude of the angular momentum and almost instantaneous reduction of the angular velocity to zero. During start up and stopping, the control moment has the maximum possible value. In the phase between start up and stopping, rotation of the spacecraft occurs with a constant (maximum admissible) value of the angular momentum, and the control moment is determined from the condition that the motion of the spacecraft about its centre of mass should strictly follow the assigned rotation trajectory specified by the calculated turn vector. The structure of the synthesized control is comparatively simple, and it can easily be implemented by vehicle-borne spacecraft motion control systems. The key parameters of the control functions that determine the optimality of the motion, according to the criterion selected, are calculated by the system described in Ref. 21 The algorithm developed for controlling spatial spacecraft reorientation enables us to apply this method in practice, since the moments of inertia of real spacecraft are similar to those of bodies with axial symmetry.

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